Arbitrage Free Generation of Potion Bonding Curves Using the Kelly Criterion

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Abstract

The Kelly Criterion allows for the calculation of options premiums where the premium is a function of capital utilization (a bonding curve)[1][2]. One important property to check for a set of options premiums is whether the premiums allow for arbitrage. In this paper, a basic set of arbitrage conditions are tested for the Kelly premiums output from a historical return distribution of Ethereum. The premiums output from the Kelly Criterion using this distribution are shown to allow arbitrage in some cases. Presented here is a method for using a set of arbitrage conditions to correct the premium prices from the Kelly Criterion so that arbitrage is no longer allowed.

1 Problem Statement

When generating a Potion Bonding Curve using the Kelly Formula (Kelly Curve) using an option-based payoff such as a Put, Call, or spread the generation process is based on the statistical properties of a returns distribution. Primarily, the entropy properties of the distribution determine how risk and reward are balanced by the Kelly formula.

As a result, the outputs of the formula may or may not conform to the no-arbitrage expectations of the option pricing problem. In other words, the prices produced may in instances violate Put-Call parity or enable the buying and selling of combinations of options which result in a risk-free payoff. Whether the prices violate these conditions depends on the probability distribution which is supplied to the Kelly formula.

If the probability distribution is derived from market data like historical prices or implied from currently observed option prices, a person analyzing the Kelly formula outputs will have control over only some aspects of this distribution. The analyzer will be able to control aspects such as sample rate, however, they will not control whether the prices generated by the distribution from real-life data will violate any arbitrage conditions. Presented here are tests for analyzing whether a set of Kelly curves conform to a loose set of no-arbitrage assumptions about the option prices.

In addition, a procedure is introduced for modifying the Kelly curve generation process to produce suboptimal curves which conform to the supplied no-arbitrage constraints. The same procedure can be utilized with any constraints more (or less) complex than the loose assumptions presented here without any loss of generality.

2 Notation

In the equations derived in the subsequent sections, the following notation will be used:

- $r \rightarrow$ risk free interest rate
- $q \rightarrow$ continuous dividend interest rate
- $C_i \rightarrow$ Call price i
- $P_i \rightarrow$ Put price i
- $K_i \rightarrow$ Strike price i
- $t \rightarrow$ current point in time

- $T_j \rightarrow$ expiration time j
- $\tau_j = (T_j - t) \rightarrow$ time until expiration j
- $S \rightarrow$ current price of the underlying

The Put-Call Parity formula used in the subsequent derivations is the version with continuous dividend equivalent to an interest rate of the underlying asset:

$$C - P = Se^{-q\tau} - Ke^{-r\tau}, \tag{1}$$

which allows the constraints to include any cash flow from the underlying that may include a dividend, yield, staking reward, or otherwise.

3 No-Arbitrage Constraints

The derivation of the no-arbitrage constraints presented here follows the derivation in Matthias Fengler's, ”Arbitrage-free smoothing of the implied volatility surface.” [3] The constraints on prices presented in Fengler are reused here as a test of the output Kelly curves. Another more complex set of constraints could be tested using the same technique.[4]

For options pricing, a certain implied volatility value maps to a certain price. As a result, an implied volatility surface (values across strikes and expirations) translates into a set of prices across strikes and expirations. If this implied volatility surface is free of arbitrage, it also means that the options prices are free of arbitrage. Fengler exploits this fact to simplify the constraints defined. If these constraints were placed on the implied volatility surface, the formulas become complex and nonlinear. In contrast, the constraints in terms of prices are much simpler and make the procedures presented here clearer for the reader.

Fengler covers only Call prices, so those will be examined first, and the Put formulas will be derived using the Call constraints and the Put-Call Parity formula. Some of these constraints on the Call prices are given in terms of first or second derivatives of the option prices. To provide checks in terms of option prices and not checks on their derivative values, these formulas are translated here to be in terms of the option price from the Kelly curve which needs to be checked.

3.1 Calls: Monotonic Constraints

One of the requirements for the Call prices to be free of arbitrage is that the prices need to be monotonic over the strikes. If the prices are not monotonic, it is possible using a vertical spread to buy an option of one strike and sell the option on another strike for risk-free profit.

Fengler derives these two constraints using the definition of the Call price and taking the derivative with respect to the strikes. This yields:

$$-e^{-r\tau} \leq \frac{C_{i+1} - C_i}{K_{i+1} - K_i} \leq 0, \tag{2}$$

note that when the interest rate r is zero or the option is at expiration, the left side simplifies to -1. These constraints will now be separated into upper and lower bound.

3.1.1 Lower Bound

Looking at the inequality for the lower bound:

$$-e^{-r\tau} \leq \frac{C_{i+1} - C_i}{K_{i+1} - K_i}, \tag{3}$$

multiplying both sides by $K_{i+1} - K_i$ we get:

$$-e^{-r\tau}\left(K_{i+1} - K_i\right) \leq C_{i+1} - C_i, \tag{4}$$

adding C_i to both sides:

$$\boxed{C_i - e^{-r\tau}\left(K_{i+1} - K_i\right) \leq C_{i+1},} \tag{5}$$

which gives the formula for the lower bound on C_{i+1}.

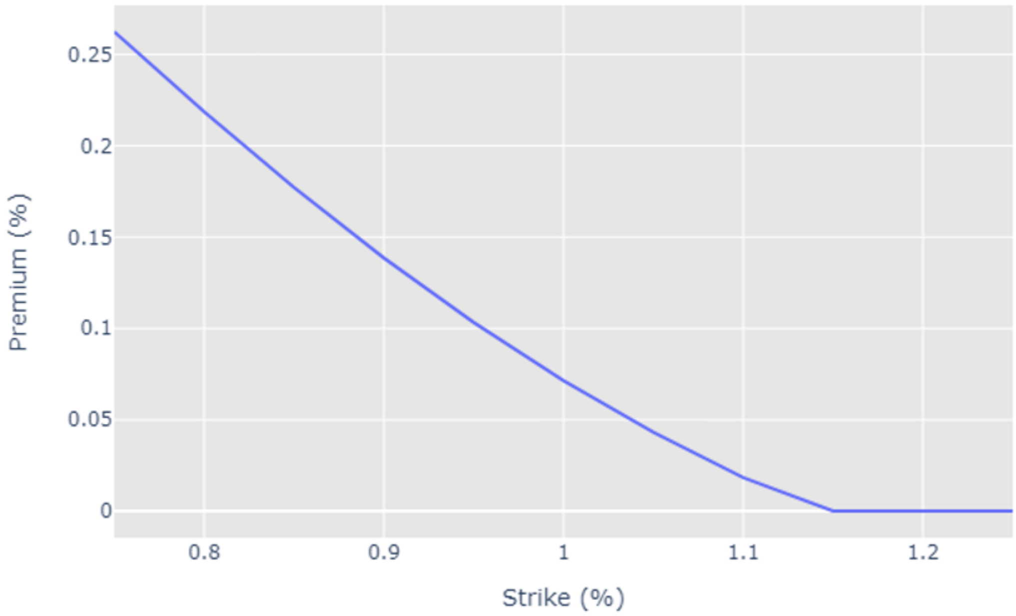

Figure 1: A plot of monotonically decreasing Call prices

3.1.2 Upper Bound

Now examining the upper bound:

$$\frac{C_{i+1} - C_i}{K_{i+1} - K_i} \leq 0, \tag{6}$$

multiplying both sides by $K_{i+1} - K_i$ we get:

$$C_{i+1} - C_i \leq 0, \tag{7}$$

and adding C_i:

$$\boxed{C_{i+1} \leq C_i,} \tag{8}$$

3.2 Calls: Convexity Constraints

The convexity of the option prices comes from the second derivative calculated with respect to the strike price always being positive. The Black-Scholes formula assumptions cause this second derivative to be equal to the risk-neutral transition density multiplied by a discounting term.[3] This is known as the Breeden and Litzenberger formula.[5] Since probabilities are always positive, the second derivative is always positive and this gives the option prices the distinctive convex curve. Note: The image contains the option prices, not the payoffs.

First, the simple lower bound:

$$\boxed{C_i \geq 0.} \tag{9}$$

Next, we take the fact that Put prices are also positive and Put-Call Parity to derive a second constraint:

$$P_i \geq 0, \tag{10}$$

use Put-Call Parity to substitute:

$$C_i - Se^{-q\tau} + Ke^{-r\tau} \geq 0, \tag{11}$$

add S and K terms to both sides:

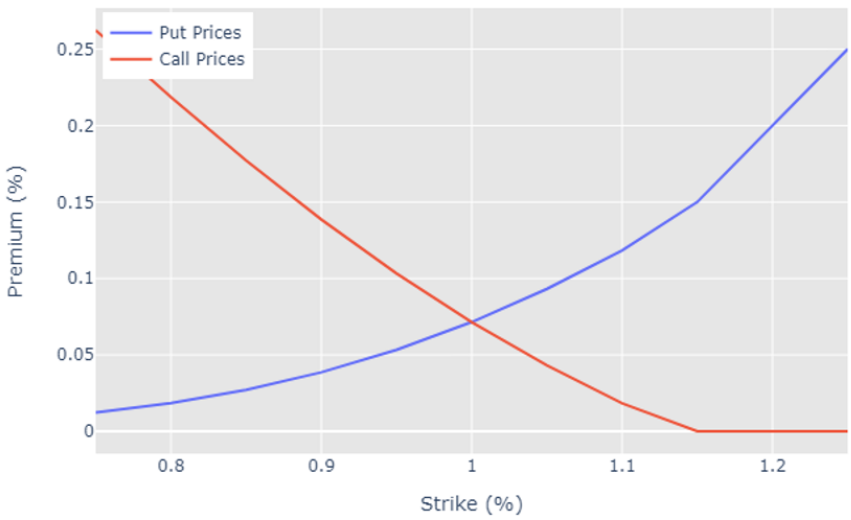

Figure 2: A plot of Call and Put prices with time still remaining before expiration at strikes 75%-125% every 5%. Note the convexity (curvature) of both price curves.

$$\boxed{C_i \geq Se^{-q\tau} - Ke^{-r\tau}.} \tag{12}$$

Finally, we have the constraint that the second derivative is positive:

$$\frac{\frac{C_{i+2}-C_{i+1}}{K_{i+2}-K_{i+1}} - \frac{C_{i+1}-C_i}{K_{i+1}-K_i}}{K_{i+2}-K_{i+1}} \geq 0, \tag{13}$$

multiply both sides by $K_{i+2} - K_{i+1}$:

$$\frac{C_{i+2}-C_{i+1}}{K_{i+2}-K_{i+1}} - \frac{C_{i+1}-C_i}{K_{i+1}-K_i} \geq 0, \tag{14}$$

and add the second term to both sides:

$$\frac{C_{i+2}-C_{i+1}}{K_{i+2}-K_{i+1}} \geq \frac{C_{i+1}-C_i}{K_{i+1}-K_i}, \tag{15}$$

which is consistent with our expectation that if the second derivative is positive, the slope should be growing as we iterate over the strikes. Next, multiply both sides by $K_{i+2} - K_{i+1}$

$$C_{i+2} - C_{i+1} \geq \frac{C_{i+1}-C_i}{K_{i+1}-K_i} \left(K_{i+2} - K_{i+1}\right), \tag{16}$$

and finally add C_{i+1} to both sides.

$$\boxed{C_{i+2} \geq C_{i+1} + \frac{C_{i+1}-C_i}{K_{i+1}-K_i} \left(K_{i+2} - K_{i+1}\right).} \tag{17}$$

Finally, the Call upper bound[3]:

$$\boxed{C_i \leq Se^{-q\tau}} \tag{18}$$

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3.3 Calls: Calendar Constraints

Calendar arbitrage is when mispricing exists across expirations and a trader can use calendar spreads and other time-based spreads to obtain a risk-free profit. This is one of the trickiest conditions because the IV surface is allowed in certain situations to be downward sloping or humped. This downward sloping in the IV values can occur despite the option price monotonically increasing at each expiration further in time. However, as long as the total variance is strictly increasing for each expiration the prices will be free of calendar arbitrage.[3] This condition gives the constraint:

$$\frac{C_{m+1}e^{\int_0^{T_{m+1}} r\,\mathrm{d}t}}{K} > \frac{C_m e^{\int_0^{T_m} r\,\mathrm{d}t}}{K}, \tag{19}$$

where C_m and T_m are the Call price and time of expiration m. First, since r is a constant here, let us remove it from the integral:

$$\frac{C_{m+1}e^{r\int_0^{T_{m+1}} \mathrm{d}t}}{K} > \frac{C_m e^{r\int_0^{T_m} \mathrm{d}t}}{K}. \tag{20}$$

Using the Fundamental Theorem of Calculus, we solve the integrals:

$$\frac{C_{m+1}e^{rT_{m+1}}}{K} > \frac{C_m e^{rT_m}}{K}, \tag{21}$$

and multiply both sides by K:

$$C_{m+1}e^{rT_{m+1}} > C_m e^{rT_m}, \tag{22}$$

then, divide both sides by the discounting term:

$$C_{m+1}e^{rT_{m+1}}e^{-rT_m} > C_m, \tag{23}$$

and use exponent addition rules to get:

$$\boxed{C_{m+1}e^{r(T_{m+1}-T_m)} > C_m.} \tag{24}$$

This constraint is consistent with our intuition because the Call price at the next expiration is multiplied by a discounting term for the interest in the period between the two expirations.

3.4 Puts: Monotonic Constraints

Using the Put-Call Parity formula, we can now derive the same constraints for Puts.

3.4.1 Lower Bound

Starting with the equivalent Call condition:

$$C_i - e^{-r\tau}\left(K_{i+1} - K_i\right) \leq C_{i+1}, \tag{25}$$

substituting using PCP:

$$Se^{-q\tau} - K_i e^{-r\tau} + P_i - e^{-r\tau}\left(K_{i+1} - K_i\right) \leq Se^{-q\tau} - K_{i+1}e^{-r\tau} + P_{i+1}, \tag{26}$$

subtracting the S term from both sides, and expanding:

$$P_i - K_i e^{-r\tau} + K_i e^{-r\tau} - K_{i+1}e^{-r\tau} \leq P_{i+1} - K_{i+1}e^{-r\tau}, \tag{27}$$

finally, adding the K_{i+1} term to both sides, and cancelling out the K_i terms, we get:

$$\boxed{P_i \leq P_{i+1}.} \tag{28}$$

3.4.2 Upper Bound

Again starting with the equivalent Call condition:

$$C_{i+1} \leq C_i, \tag{29}$$

substituting using PCP:

$$Se^{-q\tau} - K_{i+1}e^{-r\tau} + P_{i+1} \leq Se^{-q\tau} - K_i e^{-r\tau} + P_i, \tag{30}$$

and subtracting the S term from both sides:

$$P_{i+1} - K_{i+1}e^{-r\tau} \leq P_i - K_i e^{-r\tau}, \tag{31}$$

and finally adding the K_{i+1} term to both sides:

$$\boxed{P_{i+1} \leq P_i - K_i e^{-r\tau} + K_{i+1}e^{-r\tau}.} \tag{32}$$

3.5 Puts: Convexity Constraints

Beginning with the Call condition:

$$C_{i+2} \geq C_{i+1} + \frac{C_{i+1} - C_i}{K_{i+1} - K_i}\left(K_{i+2} - K_{i+1}\right), \tag{33}$$

substituting using PCP:

$$\begin{aligned} Se^{-q\tau} - K_{i+2}e^{-r\tau} + P_{i+2} \geq Se^{-q\tau} - K_{i+1}e^{-r\tau} + P_{i+1} + \\ \frac{Se^{-q\tau} - K_{i+1}e^{-r\tau} + P_{i+1} - Se^{-q\tau} + K_i e^{-r\tau} - P_i}{K_{i+1} - K_i}\left(K_{i+2} - K_{i+1}\right), \end{aligned} \tag{34}$$

subtracting the S term from both sides, and simplifying:

$$P_{i+2} - K_{i+2}e^{-r\tau} \geq P_{i+1} - K_{i+1}e^{-r\tau} + \frac{P_{i+1} - P_i - e^{-r\tau}\left(K_{i+1} - K_i\right)}{K_{i+1} - K_i}\left(K_{i+2} - K_{i+1}\right), \tag{35}$$

add the K_{i+2} to both sides and use the distributive law:

$$P_{i+2} \geq P_{i+1} + e^{-r\tau}\left(K_{i+2} - K_{i+1}\right) + \frac{P_{i+1} - P_i - e^{-r\tau}\left(K_{i+1} - K_i\right)}{K_{i+1} - K_i}\left(K_{i+2} - K_{i+1}\right), \tag{36}$$

use the distributive law again for $K_{i+2} - K_{i+1}$:

$$P_{i+2} \geq P_{i+1} + \left[e^{-r\tau} + \frac{P_{i+1} - P_i}{K_{i+1} - K_i} - \frac{e^{-r\tau}\left(K_{i+1} - K_i\right)}{K_{i+1} - K_i}\right]\left(K_{i+2} - K_{i+1}\right), \tag{37}$$

cancel out terms:

$$\boxed{P_{i+2} \geq P_{i+1} + \frac{P_{i+1} - P_i}{K_{i+1} - K_i}\left(K_{i+2} - K_{i+1}\right),} \tag{38}$$

which mirrors our Call convexity constraint. Next, we have the constraints from the requirement that Puts and Calls must have positive prices:

$$\boxed{P_i \geq 0,} \tag{39}$$

and through Put-Call Parity:

$$\boxed{P_i \geq K_i e^{-r\tau} - Se^{-q\tau}.} \tag{40}$$

Finally, from the Call constraint:

$$C_i \leq Se^{-q\tau}, \tag{41}$$

and substituting using Put-Call Parity:

$$Se^{-q\tau} - K_i e^{-r\tau} + P_i \leq Se^{-q\tau}, \tag{42}$$

and rearranging:

$$\boxed{P_i \leq K_i e^{-r\tau}} \tag{43}$$

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3.6 Puts: Calendar Constraints

For the Put calendar constraint, we begin with the Call calendar constraint:

$$C_{m+1} e^{r(T_{m+1}-T_m)} > C_m, \tag{44}$$

substituting using PCP:

$$e^{r(T_{m+1}-T_m)} \left[Se^{-q\tau} - Ke^{-r\tau} + P_{m+1}\right] > Se^{-q\tau} - Ke^{-r\tau} + P_m, \tag{45}$$

subtract $Se^{-q\tau}$ and add $Ke^{-r\tau}$ to both sides, and group like terms:

$$\boxed{Se^{-q\tau}\left[e^{r(T_{m+1}-T_m)} - 1\right] + Ke^{-r\tau}\left[1 - e^{r(T_{m+1}-T_m)}\right] + P_{m+1} e^{r(T_{m+1}-T_m)} > P_m.} \tag{46}$$

4 Experimental Analysis

This section details how to use the conditions derived in the previous section to check for arbitrage violations in the price of Kelly curves. After checking whether arbitrage violations exist, a procedure is detailed for using the constraints to calculate suboptimal Kelly curves which comply with the supplied no-arbitrage conditions.

4.1 Arbitrage Analysis

First, the Kelly curves were calculated for selling a range of Puts across different strikes, expirations, and utilization (bet fraction in Kelly terms) parameters for Ethereum (i.e. the space of arguments):

- Expirations (days): 1, 7, 14, 21
- Strikes (%): 75, 80, 85, 90, 95, 100, 105, 110, 115, 120, 125
- Util (%): 0-99.9 every 2 percent
- Training Data Range (Daily Prices): August 8th, 2015 to November 30th, 2021

All of the arbitrage constraints were tested at each point in the space of arguments in Figure 3. Each testing result corresponds to one circle in the graph: green for no-arbitrage and red for at least one arbitrage constraint is violated.

In this case, the deeply in-the-money strikes violate two of the no-arbitrage constraints for Puts. This can be seen in the region where the price graphs for each expiration cross. The two constraints which are violated are the Calendar Constraint (farther expirations should be more premium than closer expirations), and the Put-Call parity constraint in Equation 40 requiring Calls to be a positive value. In other words, at the prices of those Puts, the equivalent out-of-the-money Calls would need to be at a price less than zero.

It can be seen in Figure 4 and Figure 5 that the violations are not as severe at higher utils. The reason for this is because the no-arbitrage boundary which was derived from the definition of a Call does not depend on capital utilization. As premiums are higher at higher utilization, the prices are closer to the boundary.

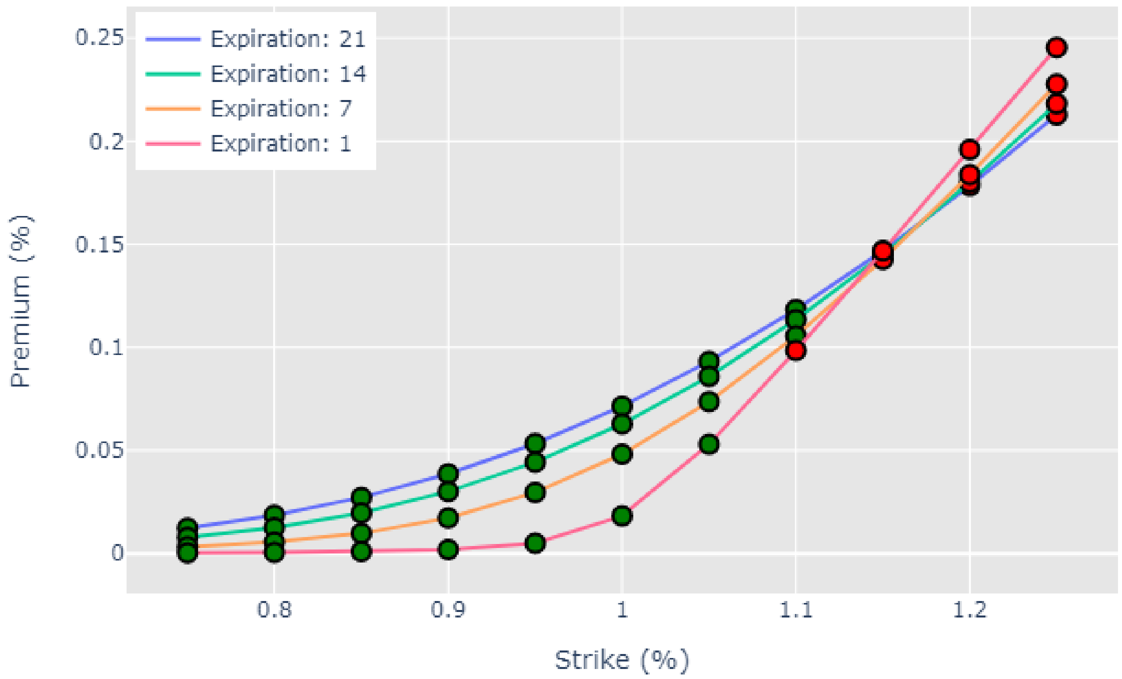

Figure 3: Put prices at each strike and expiration for util at 30%.

Since the no-arbitrage constraints would calculate a higher premium, from the Kelly perspective the Put seller would be under-betting according to the supplied probability distribution. Both of these constraints are lower bounds, so by calculating additional premium above what the Kelly formula is asking according to the distribution it would not cause the Put seller to be in danger of negative expected return over time. In the next section, the Kelly curves are compared with curves generated by constraining the Kelly curves using the arbitrage conditions.

4.2 Generating No-Arbitrage Kelly Curves

Some users may wish to calculate suboptimal Kelly curves which are consistent with a defined set of no-arbitrage constraints. For this process, the no-arbitrage constraints are used to confine the range of premiums in which the Kelly optimizer is running. The true Kelly optimum may lie outside the range of premiums being searched, and instead, the optimizer will select the no-arbitrage boundary value which is the best it can find.

Some of the results of this process can be seen in Figure 6 and Figure 7. This curve comparison is at a strike that is deeply in-the-money where it was observed the constraint violations were happening. In curves at strikes where there is no violation, the optimal and no-arbitrage Kelly curves are equivalent. As a result, they are not discussed further.

Note that at higher utils, the two curves in Figure 6 and Figure 7 are above the no-arbitrage value and the two curves are the same. This is consistent with the output seen in Figure 4 and Figure 5 where the boundary violations were not as numerous for higher capital utilization comparisons. There are only one or two results observed where at high utils the entire Kelly curve was below the constraint.

The No-Arbitrage Kelly Put prices from the suboptimal curves can be seen in Figure 8 and Figure 9. The constraints hold at every util (bet fraction), and the price curve crossing behavior which was seen in the earlier analysis is not present. This process can be generalized with any input return distribution for curve generation.

5 Conclusion

No-arbitrage constraints were derived for both Calls and Puts. These loose constraints were then used to check a set of Kelly curves derived from daily historical Ethereum returns for violations of the arbitrage as-

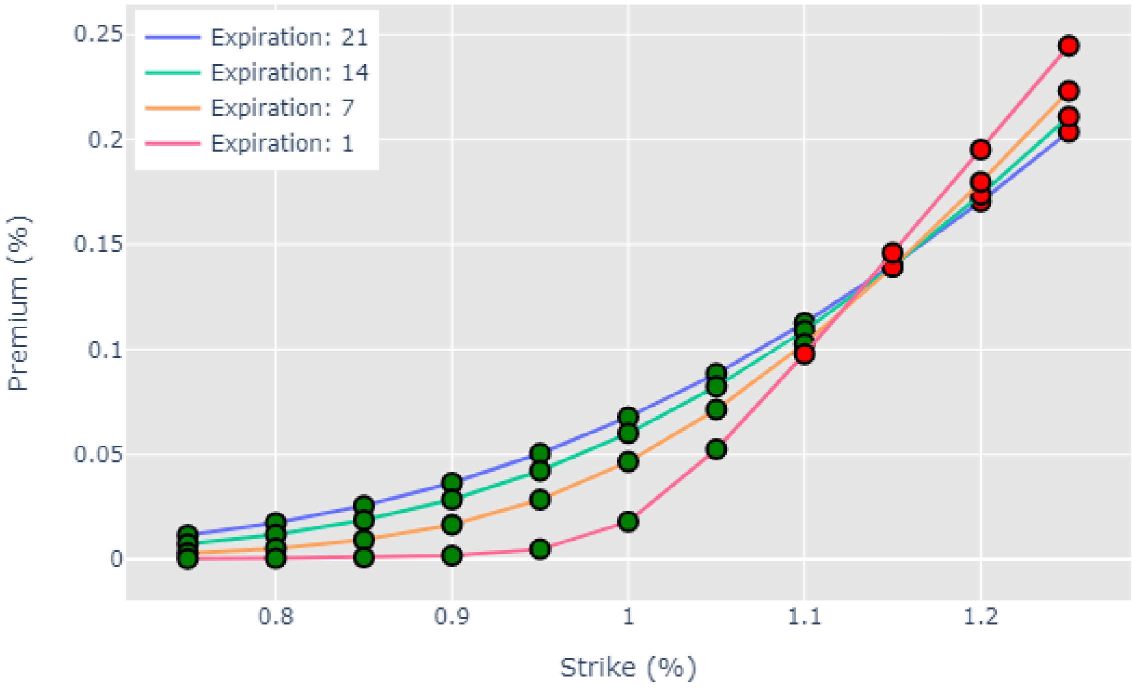

Figure 4: Put prices at each strike and expiration for util at 10%.

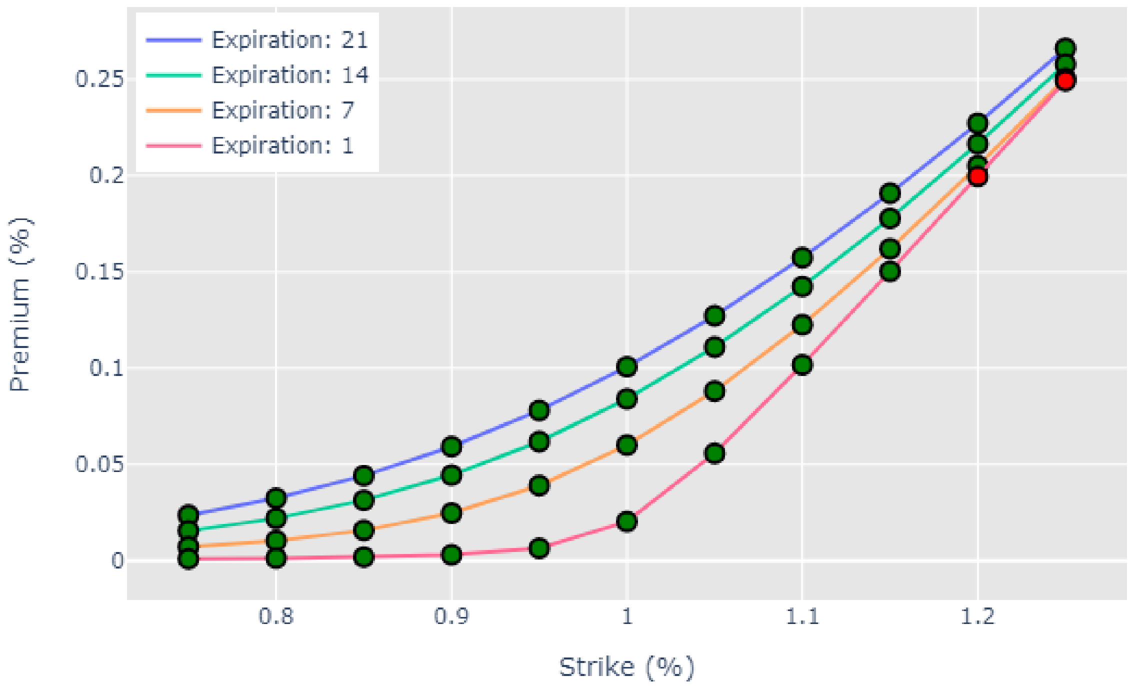

Figure 5: Put prices at each strike and expiration for util at 99.9%.

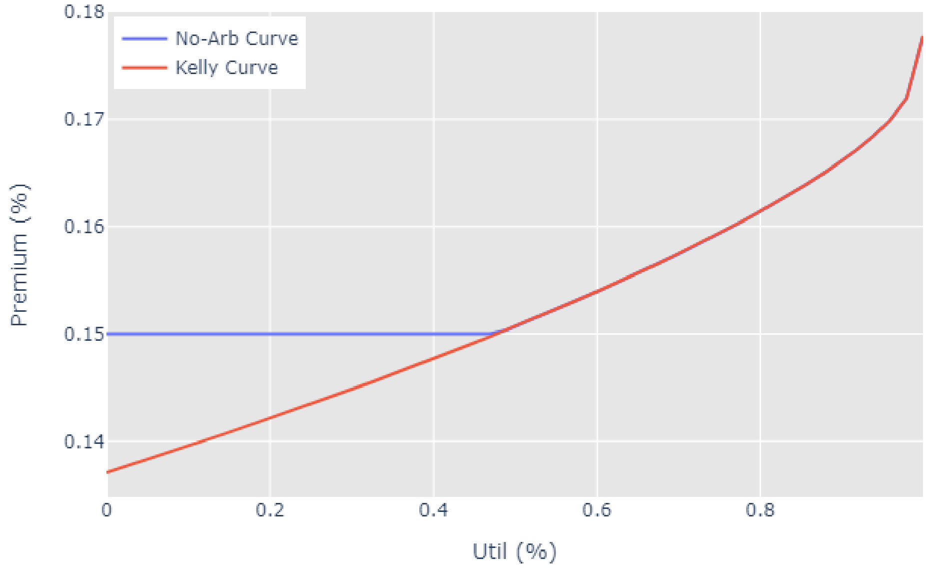

Figure 6: The Kelly Curve (Red) vs the suboptimal Kelly Curve (Blue) which obeys the no-arbitrage constraints for 115% ITM and 14 Days to Expiration.

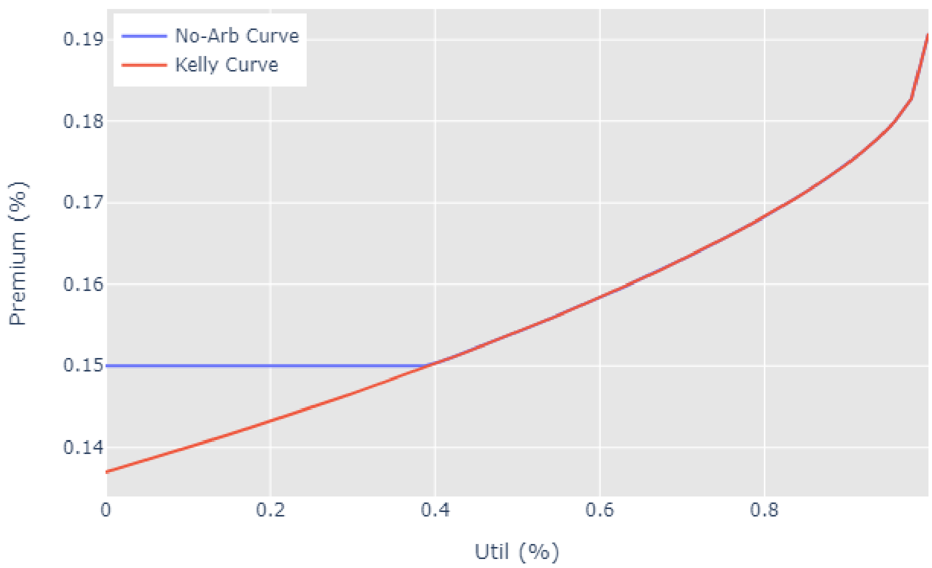

Figure 7: The Kelly Curve (Red) vs the suboptimal Kelly Curve (Blue) which obeys the no-arbitrage constraints for 115% ITM and 21 Days to Expiration.

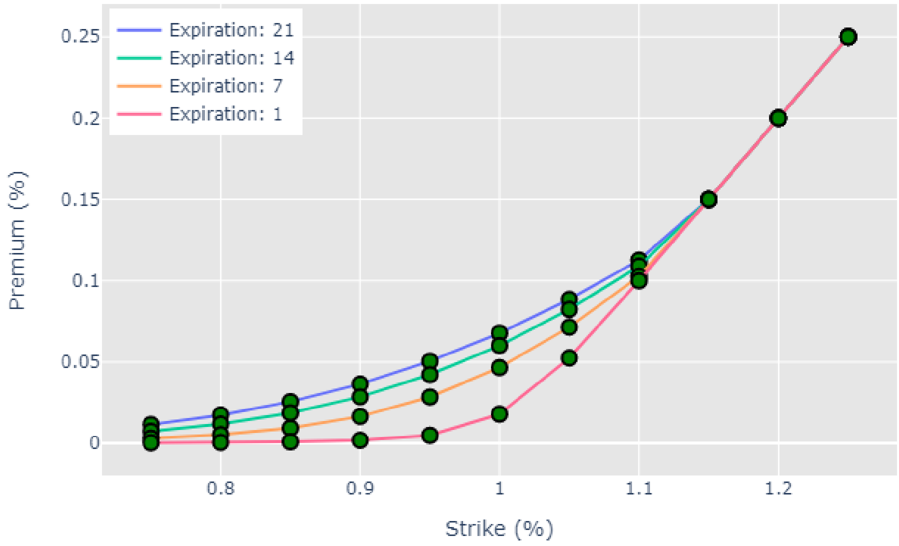

Figure 8: No-Arbitrage Kelly Put prices at each strike and expiration for util at 10%.

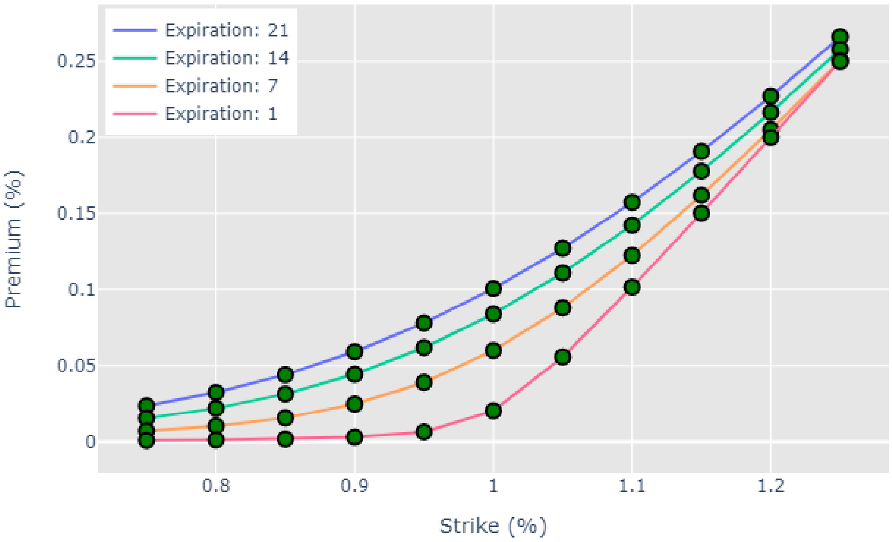

Figure 9: No-Arbitrage Kelly Put prices at each strike and expiration for util at 99.9%.

sumptions and Put-Call parity. After demonstrating the prices generated for this data violate the constraints in certain cases, it was shown that the no-arbitrage boundaries can be used to constrain the optimizer. The Kelly curves generated by the constrained optimizer were in certain cases suboptimal but maintained consistency with the defined no-arbitrage boundaries. This method can be used with any constraint set, probability distribution, or payoff function and the Kelly formula without any loss of generality.

References

[1] ”Kelly Optimal Bonding Curves”.
https://www.notion.so/Kelly-Optimal-Bonding-Curves-81c597d68338416683c025dcbe906740

[2] Johnson, Mosquera, Muñoz. ”Potion Bonding Curve Generation for Fat-Tailed Models Using the Kelly Criterion”.
https://www.notion.so/Bonding-Curve-Generation-for-Fat-Tailed-Models-c6293acf1f724d898877a41cf8455052

[3] Fengler, Matthias R. ”Arbitrage-free smoothing of the implied volatility surface.” Quantitative Finance 9.4 (2009): 417-428.

[4] Gatheral, Jim, and Antoine Jacquier. ”Arbitrage-free SVI volatility surfaces.” Quantitative Finance 14.1 (2014): 59-71.

[5] Breeden, Douglas T., and Robert H. Litzenberger. ”Prices of state-contingent claims implicit in option prices.” Journal of business (1978): 621-651.